

ABOUT THE DIOPHANTINE EQUATION $x^4 - q^4 = py^r$
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ABSTRACT:

In this paper, we prove a theorem about the integer solutions to the Diophantine equation $x^4 - q^4 = py^r$, extending previous work of K.Győry, and F.Luca and A.Togbe, and of the author.

MSC (2000): 11D41

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1 Introduction

Many Diophantine equations have been studied connected with the one in the title. For example, Kálmán Győry studied (in [4],[5]) the Diophantine equation $x^p + y^p = cz^p$. B. Powell and Henri Darmon studied the Diophantine equation $x^4 - y^4 = z^p$.

In [10] B. Powell proved that this equation has no integer solutions with p does not divide xyz . In [3] H. Darmon obtained (using elliptic curves) the following result:

Let $p \geq 3$ be a prime. Then:

- i) *the Diophantine equation $x^4 - y^4 = z^p$ has no nontrivial solutions if $p \equiv 1 \pmod{4}$.*
- i) *the Diophantine equation $x^4 - y^4 = z^p$ has no nontrivial solutions with z even.*

In some previous papers ([12],[13],[14]) we considered some Diophantine equations of the form $x^4 - q^4 = py^r$, with $r \in \{3, 5, 7\}$, where p, q are distinct prime natural numbers satisfying some conditions. Florian Luca and Alain Togbe have recently studied the equation from the title in the case $r = 3$. In [9], they showed that the Diophantine equation $x^4 - q^4 = py^3$ has no integer solutions (x, y, p, q) with $\gcd(x, y) = 1$, $xy \neq 0$, and p and q primes.

Here we try to generalize the results from the papers [12],[13], taking r a prime natural number different from p and q , all of them satisfying conditions which will be given.

The main result in this paper is:

Main Theorem. *Let p, q, r be distinct prime numbers satisfying the conditions:
 $q \neq 2$, $p \equiv 3 \pmod{4}$, $p \equiv 1 \pmod{r}$, $r \equiv \pm 3 \pmod{8}$, \bar{p} is a generator of the group
 $(U(\mathbb{Z}_{q^{r-1}}), \cdot)$, \bar{q} is a generator of the group (\mathbb{Z}_r^*, \cdot) , 2 is an r -power residue mod q . Then,
any solution in coprime integers (x, y) to the equation*

$$x^4 - q^4 = py^r$$

satisfies the property that p must divide y .

2 Preliminaries

The proofs involve techniques based on the theory of Kummer fields and cyclotomic fields. For convenience sake which we recall in this section those properties of ideals in integer rings of such fields which we will be using in our proofs.

For a prime number p and for ζ a primitive root of order l of unity (where $\gcd(p, l) = 1$), we have the following proposition:

Proposition 2.1. ([2]). *Let $l \geq 3$ be a positive integer and ζ a primitive l -th root of unity. Let $\mathbb{Z}[\zeta]$ be the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta)$. If p is a prime natural number, l is not divisible by p , and f is the smallest positive integer such that $p^f \equiv 1 \pmod{l}$, then we have*

$$p\mathbb{Z}[\zeta] = P_1 P_2 \dots P_r,$$

where $r = \frac{\varphi(l)}{f}$, P_j , $j = 1, 2, \dots, r$ are different prime ideals in the ring $\mathbb{Z}[\zeta]$.

For the ring of integers in the Kummer field $\mathbb{Q}(\sqrt[l]{a}; \zeta)$, where $a \in \mathbb{Z}$, the ideal PA , with $P \in \text{Spec}(\mathbb{Z}[\zeta])$, is totally characterized by the l power-character $\left\{ \frac{a}{P} \right\}$, as in the following theorem.

Theorem 2.2. ([7], [8]). *Let l be a prime number, ζ a primitive l -th root of unity and A the ring of integers of the Kummer field $\mathbb{Q}(\sqrt[l]{a}; \zeta)$, where $a \in \mathbb{Z}$, and P be a prime ideal in the ring $\mathbb{Z}[\zeta]$. Then the following statements hold:*

- i) *The ideal PA is equal with the l -power of a prime ideal in the ring A , if $\left\{ \frac{a}{P} \right\} = 0$.*
- ii) *The ideal PA decomposes in l different prime ideals in the ring A , if $\left\{ \frac{a}{P} \right\} = 1$.*
- iii) *The ideal PA is a prime ideal in the ring A , if $\left\{ \frac{a}{P} \right\}$ equals a root of order l of unity, different from 1.*

Now we recall properties of some Galois extensions of Kummer or cyclotomic types.

Proposition 2.3.([11]). *Let A be the ring of integers of the Kummer field $\mathbb{Q}(\sqrt[l]{a}; \zeta)$, where a is a positive integer and let ζ be a primitive l -th root of unity. Let G be the Galois group of the Kummer field $\mathbb{Q}(\sqrt[l]{a}; \zeta)$ over $\mathbb{Q}(\zeta)$. Then, for any $\sigma \in G$ and for any $P \in \text{Spec}(A)$, we have $\sigma(P) \in \text{Spec}(A)$.*

Proposition 2.4.([11]). *Let be given an extension of fields $\mathbb{Q} \subset \mathbf{K} = \mathbb{Q}(\zeta, \sqrt[n]{a})$, where ζ is a primitive n -th root of unity. Then the extension $\mathbb{Q} \subset \mathbf{K}$ is a Galois extension, the Galois group $G(\mathbf{K}/\mathbb{Q})$ is solvable and the Galois group $G(\mathbf{K}/\mathbb{Q}(\zeta))$ is cyclic.*

Theorem 2.5.([7],[8]). *Let $n \in \mathbb{N}$, $n \geq 2$, and $\mathbb{Q} \subset \mathbf{K}$ be an extension of fields, $[\mathbf{K} : \mathbb{Q}] = n$ and p be a prime number. Let \mathbb{Z}_K be the ring of integers of the field K . There exist positive integers e_i , $i = 1, 2, \dots, g$, such that*

$$p\mathbb{Z}_K = \prod_{i=1}^g P_i^{e_i},$$

where all P_i , $i = 1, 2, \dots, g$, are prime ideals above p .

The integer e_i is called the **ramification index** of p at P_i . The degree f_i of the field extension defined by

$$f_i = [\mathbb{Z}_K/P_i : \mathbb{Z}/p\mathbb{Z}]$$

is called the **residual degree** of p .

Theorem 2.6.([7],[8]). *We have the following formulas:*

$$N(P_i) = p^{f_i}, \text{ and } \sum_{i=1}^g e_i f_i = n = [\mathbf{K} : \mathbb{Q}].$$

In the case when $\mathbb{Q} \subset \mathbf{K}$ is a Galois extension, the result is more specific: the ramification indices e_i of P_i $i = 1, 2, \dots, g$, are equal (say to e), the residual degrees f_i are equal as well (say to f) and $efg = n$.

3 Results

Now we consider the Diophantine equation

$$(3.1) \quad x^4 - q^4 = py^r$$

in the conditions

$$(3.2) \quad p, q, r \text{ are primes, } p \neq q \neq r \neq p, q \neq 2, p \equiv 3 \pmod{4}, p \equiv 1 \pmod{r},$$

$r \equiv \pm 3 \pmod{8}$, \bar{p} is a generator of the group $(U(\mathbb{Z}_{q^{r-1}}), \cdot)$, \bar{q} is a generator of the group (\mathbb{Z}_r^*, \cdot) , 2 is an r -power residue mod q .

We are working in the Kummer fields $\mathbb{Q}(\zeta; \sqrt[r]{p})$ and $\mathbb{Q}(\zeta; \sqrt[r]{2^{r-2}p})$, where p, r are prime numbers, $p \equiv 1 \pmod{r}$.

Here are some examples of primes p, q, r satisfying the conditions (3.2).

1. $p = 19, r = 3, q = 11$. We have: $19 \equiv 3 \pmod{4}$, $19 \equiv 1 \pmod{3}$, $3 \equiv 3 \pmod{8}$, $\overline{19}$ is a generator of the group $(U(\mathbb{Z}_{121}), \cdot)$, $\overline{11} = \overline{2}$ is a generator of the group (\mathbb{Z}_3^*, \cdot) , 2 is a 3-power residue mod 11.

2. $p = 67, r = 3, q = 5$. We have: $67 \equiv 3 \pmod{4}$, $67 \equiv 1 \pmod{3}$, $3 \equiv 3 \pmod{8}$, $\overline{67}$ is a generator of the group $(U(\mathbb{Z}_{25}), \cdot)$, $\overline{5} = \overline{2}$ is a generator of the group (\mathbb{Z}_3^*, \cdot) , 2 is a 3-power residue mod 5.

3. $p = 11, r = 5, q = 3$. We have: $11 \equiv 3 \pmod{4}$, $11 \equiv 1 \pmod{5}$, $5 \equiv -3 \pmod{8}$, $\overline{11}$ is a generator of the group $(U(\mathbb{Z}_{81}), \cdot)$, $\overline{3}$ is a generator of the group (\mathbb{Z}_5^*, \cdot) , 2 is a 5-power residue mod 3.

4. $p = 67, r = 11, q = 13$. We have: $67 \equiv 3 \pmod{4}$, $67 \equiv 1 \pmod{11}$, $11 \equiv 3 \pmod{8}$, $\overline{67}$ is a generator of the group $(U(\mathbb{Z}_{13^{10}}), \cdot)$, $\overline{13} = \overline{2}$ is a generator of the group $(\mathbb{Z}_{11}^*, \cdot)$, 2 is a 11-power residue mod 13.

Let A be the ring of integers of the Kummer field $\mathbb{Q}(\zeta; \sqrt[r]{p})$. We give a general lemma about two ideals generated in A by elements of the form: $y_2 - \zeta^m \sqrt[r]{p} y_1$, with $y_1, y_2 \in \mathbb{Z}$

with different exponents $m \in \mathbb{N}$.

Lemma 3.1. *Let p and r be prime integers, $p \equiv 1 \pmod{r}$ and let ζ be a primitive r -th root of unity. Let A be the ring of integers of the Kummer field $\mathbb{Q}(\zeta; \sqrt[r]{p})$, y_1 and y_2 are integers such that $\gcd(y_1, y_2) = 1$, p does not divide y_2 , $m, n \in \{0, 1, \dots, r-1\}$, $m \neq n$. If $y_2 - y_1$ is not divisible by r , then*

$$(y_2 - \zeta^m y_1 \sqrt[r]{p}) A \text{ and } (y_2 - \zeta^n y_1 \sqrt[r]{p}) A$$

are coprime ideals of A .

Proof. We suppose that $m < n$.

Let J be the ideal of A generated by $y_2 - \zeta^m y_1 \sqrt[r]{p}$ and $y_2 - \zeta^n y_1 \sqrt[r]{p}$.

Using the Fermat's Little Theorem, we have: $y_1^r \equiv y_1 \pmod{r}$ and $y_2^r \equiv y_2 \pmod{r}$. This implies that $y_2^r - p y_1^r \equiv y_2 - p y_1 \pmod{r}$. But $p \equiv 1 \pmod{r}$, therefore $y_2^r - p y_1^r \equiv y_2 - y_1 \pmod{r}$.

Using the fact that $y_2 - y_1$ is not divisible by r , it results that $y_2^r - p y_1^r$ is not divisible by r , therefore $\gcd(r, y_2^r - p y_1^r) = 1$. This implies that there exist $h, k \in \mathbb{Z}$ such that

$$h(y_2^r - p y_1^r) + k r = 1.$$

In the ring A , we have

$$y_2^r - p y_1^r = (y_2 - y_1 \sqrt[r]{p})(y_2 - y_1 \zeta \sqrt[r]{p}) \dots (y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}).$$

This implies that:

$$y_2^r - p y_1^r \in J.$$

Since

$$(y_2 - y_1 \zeta^m \sqrt[r]{p}) - (y_2 - y_1 \zeta^n \sqrt[r]{p}) = \zeta^m y_1 \sqrt[r]{p} (\zeta^{n-m} - 1) = \zeta^m y_1 \sqrt[r]{p} u_{n-m} (\zeta - 1),$$

where $u_{n-m}, \zeta^m \in U(\mathbf{Z}[\zeta]) \subset U(A)$, we obtain that $y_1 \sqrt[r]{p} (\zeta - 1) \in J$. But $\sqrt[r]{p^{r-1}} \in A$, therefore $p y_1 (\zeta - 1) \in J$. Now, $y_2 - y_1 \zeta^m \sqrt[r]{p} \in J$ and $\zeta^{n-m} \in A$. These imply that $y_2 \zeta^{n-m} - \zeta^n y_1 \sqrt[r]{p} \in J$. Using the fact that $y_2 - y_1 \zeta^n \sqrt[r]{p} \in J$, it results that $y_2 (\zeta^{n-m} - 1) \in J$. But $\zeta^{n-m} - 1 = u_{n-m} (\zeta - 1)$, with $u_{n-m} \in U(A)$. We obtain that $y_2 (\zeta - 1) \in J$.

From the hypothesis, we know that $\gcd(y_1, y_2) = 1$ and y_2 is not divisible with p . This implies that $\gcd(py_1, y_2) = 1$, therefore there exist $h_1, h_2 \in \mathbb{Z}$ such that $py_1h_1 + y_2h_2 = 1$. Multiplying the last equality by $\zeta - 1$ and using the previous relations, we obtain that $\zeta - 1 \in J$.

Since $r = u(1 - \zeta)^{r-1}$, where $u \in U(\mathbb{Z}[\zeta]) \subset U(A)$, we have $r \in J$.

From this relation and the previous ones, it results that $1 \in J$, therefore

$$(y_2 - \zeta^m y_1 \sqrt[r]{p}) A \quad \text{and} \quad (y_2 - \zeta^n y_1 \sqrt[r]{p}) A$$

are coprime ideals of A .

Analogously we may prove:

Lemma 3.2. *Let p and r be prime integers, $p \equiv 1 \pmod{r}$ and let ζ be a primitive r -th root of unity. Let A be the ring of integers of the Kummer field $\mathbb{Q}(\zeta; \sqrt[r]{2^{r-2}p})$, y_1 and y_2 integers such that $\gcd(y_1, y_2) = 1$, p does not divide $2y_2$, $m, n \in \{0, 1, \dots, r-1\}$, $m \neq n$. If $y_2 - 2^{r-2}py_1$ is not divisible by r , then*

$$(y_2 - \zeta^m y_1 \sqrt[r]{2^{r-2}p}) A \quad \text{and} \quad (y_2 - \zeta^n y_1 \sqrt[r]{2^{r-2}p}) A$$

are coprime ideals of A .

We may consider now our equation $x^4 - q^4 = py^r$.

Proof of the Main theorem. We reason by reduction to absurd. Let $(x, y) \in \mathbb{Z}^2$, $\gcd(x, y) = 1$ be a solution to the equation (3.1), with p, q, r satisfying conditions (3.2).

Suppose, by way of contradiction that p does not divide y .

We consider two cases: either x is odd or x is even.

Case 1. x is an odd number

Since q is a prime number, $q \geq 3$, we get $x^2, q^2 \equiv 1 \pmod{4}$, therefore $x^2 - q^2 \equiv 0 \pmod{4}$, $x^2 + q^2 \equiv 2 \pmod{4}$.

We denote $d = \gcd(x^2 - q^2, x^2 + q^2)$. Then $d|2x^2$ and $d|2q^2$. But $\gcd(x, y) = 1$ implies that x is not divisible by q . Therefore $d = 2$. We get either

$$x^2 - q^2 = 2^{r-1}py_1^r \quad \text{and} \quad x^2 + q^2 = 2y_2^r,$$

where $y_1, y_2 \in \mathbb{Z}$, $2y_1y_2 = y$, y_2 is an odd number, $\gcd(y_1, y_2) = 1$,
or

$$x^2 - q^2 = 2^{r-1}y_1^r \text{ and } x^2 + q^2 = 2py_2^r,$$

where $y_1, y_2 \in \mathbb{Z}$, $2y_1y_2 = y$, y_2 is an odd number, $\gcd(y_1, y_2) = 1$.

In the last case, we obtain that $p \mid (x^2 + q^2)$, in contradiction with the fact that $p \equiv 3 \pmod{4}$. It remains to study the case

$$x^2 - q^2 = 2^{r-1}py_1^r \text{ and } x^2 + q^2 = 2y_2^r.$$

By subtracting the two equations, we obtain $q^2 = y_2^r - 2^{r-2}py_1^r$.

We consider the Kummer field $\mathbb{Q}(\zeta, \sqrt[r]{2^{r-2}p})$, where ζ is a primitive r -th root of unity.

The last equality becomes:

$$q^2 = \left(y_2 - y_1 \sqrt[r]{2^{r-2}p}\right) \left(y_2 - y_1 \zeta \sqrt[r]{2^{r-2}p}\right) \dots \left(y_2 - y_1 \zeta^{r-1} \sqrt[r]{2^{r-2}p}\right)$$

But $\langle \bar{q} \rangle = (\mathbb{Z}_r^*, \cdot)$ and, by applying Proposition 2.1, we obtain that $q\mathbb{Z}[\zeta]$ is a prime ideal in the ring $\mathbb{Z}[\zeta]$.

We try to decompose the ideal (q) in the ring A . We have:

$$\left\{ \frac{2^{r-2}p}{(q)} \right\} = \left\{ \frac{2}{(q)} \right\}^{r-2} \left\{ \frac{p}{(q)} \right\}.$$

Since 2 is an r -power residue mod q , then there is $\alpha \in \mathbb{Z}[\zeta]$ such that $\alpha^r \equiv 2 \pmod{q}$, therefore $\left\{ \frac{2}{(q)} \right\} = 1$.

We obtain that $\left\{ \frac{2^{r-2}p}{(q)} \right\} = \left\{ \frac{p}{(q)} \right\}$ and we get:

$$\left\{ \frac{p}{(q)} \right\} = \zeta^c \equiv p^{\frac{N((q))-1}{r}} \pmod{q}.$$

We next calculate $N((q))$.

Since $q\mathbb{Z}[\zeta]$ is a prime ideal in the ring $\mathbb{Z}[\zeta]$ it results that $e = 1$, $g = 1$. But $efg = [\mathbb{Q}(\zeta) : \mathbb{Q}] = r - 1$, therefore $f = r - 1$. Using Theorem 2.6, we obtain $N((q)) = q^{r-1}$ and

$$\left\{ \frac{p}{(q)} \right\} = \zeta^c \equiv p^{\frac{q^{r-1}-1}{r}} \pmod{q}.$$

If $\left\{ \frac{p}{(q)} \right\} = 1$, it results $p^{\frac{q^{r-1}-1}{r}} \equiv 1 \pmod{q}$. But

$< \bar{p} > = (U(Z_{q^{r-1}}); \cdot)$ and $|U(Z_{q^{r-1}})| = q^{r-1} - q^{r-2}$, hence $q^{r-1} - q^{r-2} \mid \frac{q^{r-1}-1}{r}$. This implies that there is $j \in \mathbb{N}^*$ such that $\frac{q^{r-1}-1}{r} = j(q^{r-1} - q^{r-2})$. Therefore $q^{r-2} + q^{r-3} + \dots + q + 1 = jr q^{r-2}$. This equality is impossible, because, for $q \in \mathbb{N}^*$, $q \geq 3$, we have: $q^{r-2} + q^{r-3} + \dots + q + 1 \leq (r-1)q^{r-2} < jr q^{r-2}$.

We obtain that $\left\{ \frac{p}{(q)} \right\} = \zeta^c \neq 1$, therefore $qA \in \text{Spec}(A)$. Passing to ideals in the expression of q^2 , we get:

$$\left(y_2 - y_1 \sqrt[r]{2^{r-2}p} \right) A \left(y_2 - y_1 \zeta \sqrt[r]{2^{r-2}p} \right) A \dots \left(y_2 - y_1 \zeta^{r-1} \sqrt[r]{2^{r-2}p} \right) A = (qA)^2$$

and, according to Lemma 3.2, this equality is impossible.

Case 2. x is an even number.

In this case, $x^2 - q^2$ and $x^2 + q^2$ are odd numbers.

We prove that $\gcd(x^2 - q^2, x^2 + q^2) = 1$. We suppose that there exists an odd prime number d such that $d \mid (x^2 - q^2)$ and $d \mid (x^2 + q^2)$. Hence $d \mid x$ and $d \mid q$. Using the hypothesis, we obtain that $d \mid y$, in contradiction with the fact $\gcd(x, y) = 1$. Therefore $\gcd(x^2 - q^2, x^2 + q^2) = 1$. The equation (3.1) implies either

$$x^2 - q^2 = py_1^r, x^2 + q^2 = y_2^r, \text{ with } y_1, y_2 \in \mathbb{Z}, y_1 y_2 = y, \gcd(y_1, y_2) = 1$$

or

$$x^2 - q^2 = y_1^r, x^2 + q^2 = py_2^r, \text{ with } y_1, y_2 \in \mathbb{Z}, y_1 y_2 = y, \gcd(y_1, y_2) = 1.$$

In the last case, we obtain that $p \mid (x^2 + q^2)$, in contradiction with the fact that $p \equiv 3 \pmod{4}$. It remains the case

$$x^2 - q^2 = py_1^r \text{ and } x^2 + q^2 = y_2^r.$$

Subtracting the two equations, we get $2q^2 = y_2^r - py_1^r$.

Let $\mathbb{Q}(\zeta, \sqrt[r]{p})$ be a Kummer field, where ζ is a primitive r -th root of unity, and A the ring of integers of $\mathbb{Q}(\zeta, \sqrt[r]{p})$. In A , the last equality becomes

$$(y_2 - y_1 \sqrt[r]{p})(y_2 - y_1 \zeta \sqrt[r]{p}) \dots (y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}) = 2q^2.$$

We prove that 2 is a prime element in the ring $\mathbb{Z}[\zeta]$.

According to Proposition 2.1, $2\mathbb{Z}[\zeta] = P_1 P_2 \dots P_s$, where $s = \frac{\varphi(r)}{\text{ord}_{\mathbb{Z}_r^*}(\zeta)} = \frac{r-1}{\text{ord}_{\mathbb{Z}_r^*}(\zeta)}$; but $r \equiv 3$ or $5 \pmod{8}$ implies that 2 is not a quadratic residue modulo r . Applying the

Euler's Criterion, we obtain that $2^{\frac{r-1}{2}} \equiv -1 \pmod{r}$. Hence $\text{ord}_{\mathbb{Z}_r^*}(\bar{2}) = r - 1$.

We get that $2\mathbb{Z}[\zeta] \in \text{Spec}(\mathbb{Z}[\zeta])$

As p is a prime number which satisfies $p \equiv 3 \pmod{4}$, we have $\left\{\frac{p}{(2)}\right\} = 1$. Using Theorem 2.2, we obtain that $2A = P'_1 P'_2 \dots P'_r$, where P'_1, P'_2, \dots, P'_r are prime ideals in the ring A .

As in the case 1, $qA \in \text{Spec}(A)$. By considering ideals in the relation in the expression of $2q^2$, we obtain

$$(3.3) \quad (y_2 - y_1 \sqrt[r]{p}) A (y_2 - y_1 \zeta \sqrt[r]{p}) A \dots (y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}) A = P'_1 P'_2 \dots P'_r (qA)^2.$$

Let G be the Galois group of the Kummer field $\mathbb{Q}(\zeta, \sqrt[r]{p})$ over $\mathbb{Q}(\zeta)$. According to Proposition 2.4, G is a cyclic group with σ as a generator, $\sigma : \mathbb{Q}(\zeta, \sqrt[r]{p}) \rightarrow \mathbb{Q}(\zeta, \sqrt[r]{p})$, $\sigma(\sqrt[r]{p}) = \zeta \sqrt[r]{p}$. We come back to the equality (3.3) and we consider three cases.

Subcase (i): If there exists $k \in \{1, 2, \dots, r\}$ such that $(y_2 - y_1 \sqrt[r]{p}) A = P'_k \in \text{Spec}(A)$, we use Proposition 2.3, and we obtain that $\sigma((y_2 - y_1 \sqrt[r]{p}) A) = (y_2 - y_1 \zeta \sqrt[r]{p}) A \in \text{Spec}(A)$ and $\sigma^2((y_2 - y_1 \sqrt[r]{p}) A) = (y_2 - y_1 \zeta^2 \sqrt[r]{p}) A \in \text{Spec}(A), \dots, \sigma^{r-1}((y_2 - y_1 \sqrt[r]{p}) A) = (y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}) A \in \text{Spec}(A)$, therefore the equality (3.3) is impossible.

Subcase (ii): If $(y_2 - y_1 \sqrt[r]{p}) A = P_{11}^2$, where $P_{11} \in \text{Spec}(A)$, using Proposition 2.3, we obtain $(y_2 - y_1 \zeta \sqrt[r]{p}) A = \sigma((y_2 - y_1 \sqrt[r]{p}) A) = P_{12}^2$, where $P_{12} \in \text{Spec}(A)$, and so on, up to $(y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}) A = \sigma^{r-1}((y_2 - y_1 \sqrt[r]{p}) A) = P_{1r}^2$, where $P_{1r} \in \text{Spec}(A)$.

The equality (3.3) becomes

$$P_{11}^2 P_{12}^2 \dots P_{1r}^2 = P'_1 P'_2 \dots P'_r (qA)^2,$$

where $P_{1i} \in \text{Spec}(A)$, $(\forall) i = \overline{1, r}$ $P'_i \in \text{Spec}(A)$, $i = 1, 2, \dots, r$, $qA \in \text{Spec}(A)$. This equality is impossible, since all intervening ideals are prime in A and $r > 2$.

Subcase (iii): If $(y_2 - y_1 \sqrt[r]{p}) A = P_{21} P'_{21}$, where $P_{21}, P'_{21} \in \text{Spec}(A)$, $P_{21} \neq P'_{21}$, then $\sigma((y_2 - y_1 \sqrt[r]{p}) A) = (y_2 - y_1 \zeta \sqrt[r]{p}) A = P_{22} P'_{22}$, where P_{21}, P'_{21} are distinct prime ideals in the ring A and so on, up to we have in the end

$$\sigma^{r-1}((y_2 - y_1 \sqrt[r]{p}) A) = (y_2 - y_1 \zeta^{r-1} \sqrt[r]{p}) A = P_{2r} P'_{2r},$$

where P_{2r}, P'_{2r} are prime ideals in A .

Therefore relation (3.3) is equivalent to

$$P_{21}P'_{21}P_{22}P'_{22}\dots P_{2r}P'_{2r} = P'_1P'_2\dots P'_r(qA)^2,$$

where $P_{21}, P'_{21}, P_{22}, P'_{22}, \dots, P_{2r}, P'_{2r}, P'_1, P'_2, \dots, P'_r, qA$ are prime ideals in A .

This equality does not hold.

From subcases (i), (ii), (iii), it results that the equality (3.3) is impossible.

We get that the supposition made is false, so p must divide y .

Concluding remarks

Finally we give some examples where at least one of the conditions (3.2) is not fulfilled and Main Theorem is false.

1. $p = 17, r = 5, q = 3$. We have: $5 \equiv -3 \pmod{8}$, $\bar{3}$ is a generator of the group (\mathbb{Z}_5^*, \cdot) , 2 is a 5-power residue mod 3, but 17 is not congruent with 3 $\pmod{4}$, 17 is not congruent with 1 $\pmod{3}$, $\bar{17}$ is not a generator of the group $(U(\mathbb{Z}_{81}), \cdot)$.

In this situation we observe that $(x, y) = (5, 2)$ is a solution of the equation $x^4 - 3^4 = 17y^5$ but p does not divide y .

2. $p = 65537, r = 11, q = 257$. We have: $11 \equiv 3 \pmod{8}$, $\bar{257} = \bar{4}$ is not a generator of the group $(\mathbb{Z}_{11}^*, \cdot)$, 2 is a 11-power residue mod 257, $\bar{65537}$ is not a generator of the group $(U(\mathbb{Z}_{257^{10}}), \cdot)$, 65537 is not congruent with 3 $\pmod{4}$, 65537 is not congruent with 1 \pmod{r} .

In this situation we observe that $(x, y) = (255, -2)$ is a solution of the equation $x^4 - 257^4 = 65537y^{11}$ but p does not divide y .

3. $p = 257, r = 7, q = 17$. We have: 7 is not congruent with $\pm 3 \pmod{8}$, $\bar{17} = \bar{3}$ is a generator of the group (\mathbb{Z}_7^*, \cdot) , 2 is a 7-power residue mod 17, $\bar{257}$ is not a generator of the group $(U(\mathbb{Z}_{17^6}), \cdot)$, 257 is not congruent with 3 $\pmod{4}$, 257 is not congruent with 1 $\pmod{7}$.

In this situation we observe that $(x, y) = (15, -2)$ is a solution of the equation $x^4 - 17^4 = 257y^7$ but p does not divide y .

In the last example we have $r \equiv -1 \pmod{8}$. This situation is not covered by Main Theorem. The particular case $r = 7$ has been subject of investigation in [14]. In near future we intend to analyse equation 3.1 also in the case $r \equiv \pm 1 \pmod{8}$.

Another remark is that there is a connection between the last three examples, more

exactly, if F_n is the n -term of Fermat sequence ($F_n = 2^{2^n} + 1$), in example **1.** we have $p = 17 = F_2$; in example **3.** we have $q = 17 = F_2$, $p = 257 = F_3$; in example **2.** we have $q = 257 = F_3$, $p = 65537 = F_4$.

In the future we will study if Lemma 3.1 and Lemma 3.2 are valid not only for $p \equiv 1 \pmod{r}$ but also in more general conditions.

Another research theme is to investigate to what extent the conditions imposed on p , q , r can be relaxed, so that analogous results could be obtained in more general conditions.

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